Neville's Method. Neville's Interpolated Iteration

Higher order polynomials are constructed from previous lower order polynomials.

Consider the construction of a polyn. through $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), \ldots (x_4, f(x_4))$

Degree Zero:

$P_0(x) = f(x_0), \quad P_1(x) = f(x_1) \ldots$ Constant polynomials

$P_2(x) = f(x_2), \ldots P_4(x) = f(x_4)$

Degree one:

$$P_{0,1}(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \quad \text{Lagrange}$$

$$P_{0,1}(x) = \frac{(x - x_0) P_1(x) - (x - x_1) P_0(x)}{x_1 - x_0}. \quad \text{Neville}$$

$\vdots$

$$P_{i,i+1}(x) = \frac{(x - x_i) P_{i+1}(x) - (x - x_{i+1}) P_i(x)}{x_{i+1} - x_i} \quad \text{Neville}$$

$$P_{1,2}(x) = \frac{(x - x_1) P_2(x) - (x - x_2) P_1(x)}{x_2 - x_1}$$

$$\boxed{P_{0,1,2}(x) = \frac{(x-x_0)\,P_{1,2}(x) - (x-x_2)\,P_{0,1}(x)}{x_2-x_0}} \qquad (A1.2)$$

$$\frac{(x-x_0)}{x_2-x_0}P_{12}(x) = \frac{(x-x_0)(x-x_1)\,P_2(x)}{(x_2-x_0)(x_2-x_1)}$$

$$\boxed{-\frac{(x-x_0)(x-x_2)}{(x_2-x_0)(x_2-x_1)}P_1(x)}$$

$$-\frac{(x-x_2)}{x_2-x_0}P_{0,1}(x) = \boxed{-\frac{(x-x_2)(x-x_0)}{(x_2-x_0)(x_1-x_0)}P_1(x)}$$

$$+\frac{(x-x_2)(x-x_1)}{(x_2-x_0)(x_1-x_0)}P_0(x)$$

$$P_1(x)\left[-\frac{(x-x_0)(x-x_2)}{\cancel{(x_2-x_0)}}\left(\frac{1}{x_2-x_1}+\frac{1}{x_1-x_0}\right)\right]$$

$$\frac{\cancel{x_1}-x_0+x_2-\cancel{x_1}}{(x_2-x_1)(x_1-x_0)} = \frac{x_2-x_0}{}$$

$$= P_1(x)\left[-\frac{(x-x_0)(x-x_2)\cancel{(x_2-x_0)}}{(x_2-x_1)(x_1-x_0)\cancel{(x_2-x_0)}}\right] = P_1(x)\left[\frac{(x-x_0)(x-x_2)}{(x_1-x_2)(x_1-x_0)}\right]$$

Summarizing,

$$P_{0,1,2}(x) = \frac{(x-x_0)P_{1,2}(x) - (x-x_2)P_{0,1}(x)}{x_2-x_0}$$

$$= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}P_2(x) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}P_1(x)$$

$$+ \frac{(x-x_2)(x-x_1)}{(x_0-x_2)(x_0-x_1)}P_0(x).$$ Lagrange Polyn.

An alternative way to show that the Lagrange and Neville representation are the same polyn. is by evaluating them at the interpolation points and showing that their values are the same.

__Proof.__ Evaluating at x_0, x_1, x_2 the polyn. (N.2)

$$P_{0,1,2}(x_0) = P_{0,1}(x_0) = f(x_0) \checkmark$$

$$P_{0,1,2}(x_2) = P_{1,2}(x_2) = f(x_2) \checkmark$$

$$P_{0,1,2}(x_1) = \frac{(x_1-x_0)P_{1,2}(x_1) - (x-x_2)P_{0,1}(x_1)}{(x_2-x_0)}$$

$$= \frac{(\cancel{x_1}-x_0)f(x_1) - (\cancel{x_1}-x_2)f(x_1)}{x_2-x_0}$$

$$= \frac{(x_2-\cancel{x_0})f(x_1)}{x_2-\cancel{x_0}} = f(x_1) \checkmark$$

Proof Notice

$$P(x_i) = \frac{x_i - x_j}{x_i - x_j} P_{0,1,\dots,j-1,j+1,\dots,k} = f(x_i)$$

$$P(x_j) = f(x_j)$$

and $P(x_r) = f(x_r)$ if $r \neq i, j$.

then $P(x) = P_{0,1,2,\dots,k}(x)$ unique polyn. of degree at most k that interp these points.

Since

$$P(x_r) = \frac{(\cancel{x_r} - x_j)\, \overset{= f(x_r)}{P_{0,1,\dots,j-1,j+1,\dots,k}(x_r)} - (\cancel{x_r} - x_i)\, \overset{= f(x_r)}{P_{0,1,\dots,i-1,i+1,\dots,k}(x_r)}}{x_i - x_j}$$

$$= \frac{\cancel{(x_i - x_j)}\, f(x_r)}{\cancel{(x_i - x_j)}} = f(x_r).$$

If $j \neq 0, n$

$$P_{0,1,\ldots,n}(x_j) = \frac{x_j - x_0}{x_n - x_0} \underbrace{P_{1,2\ldots n}(x_j)}_{= f(x_j)} - \frac{x_j - x_n}{x_n - x_0} \underbrace{P_{0,1,\ldots,n-1}(x_j)}_{= f(x_j)}$$

$$= \frac{\cancel{x_j} - x_0 - \cancel{x_j} + x_n}{x_n - x_0} f(x_j) = f(x_j) \checkmark$$

$\therefore$ $P_{0,1,\ldots,n}(x)$ interpolates all the points $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$. It's the unique polynomial of degree at most n with this property.

TABLE DESCRIBING CONSTRUCTION.

$$P_{1,2}(x) = \frac{(x-x_1)P_2 - (x-x_2)P_1}{x_2 - x_1}$$

$$P_{0,1,2}(x) = \frac{(x-x_0)P_{12} - (x-x_2)P_{01}}{x_2 - x_0}$$

P_0				
P_1	$P_{0,1}$			
P_2	$P_{1,2}$	$P_{0,1,2}$		
P_3	$P_{2,3}$	$P_{1,2,3}$	$P_{0,1,2,3}$	
P_4	$P_{3,4}$	$P_{2,3,4}$	$P_{1,2,3,4}$	$P_{0,1,2,3,4}$.

$$P_{0,1,2,3,4}(x) = \frac{(x-x_0)P_{1,2,3,4} - (x-x_4)P_{0,1,2,3}}{x_4 - x_0}$$

Above table can be improved Written as an actual array as

Note: This is a good exercise to discuss programming with new students. Work on code

$f(x_0) = P_0 = Q_{0,0}$				
$f(x_1) = P_1 = Q_{1,0}$	$P_{0,1} = Q_{11}$			
$f(x_2) = P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$Q_{22} = P_{0,1,2}$		
$f(x_3) = P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$Q_{32} = P_{1,2,3}$	$Q_{33} = P_{0,1,2,3}$	
$f(x_4) = P_4 = Q_{4,0}$	$P_{3,4} = Q_{4,1}$	$Q_{42} = P_{2,3,4}$	$Q_{43} = P_{1,2,3,4}$	$Q_{44} = P_{0,1,2,3,4}$

Def.– Q_{ij} is polyn. of degree at most j interpolating points $(x_{i-j}, f(x_{i-j})), (x_{i-j+1}, f(x_{i-j+1})), \ldots, (x_i, f(x_i))$

$j \le i$

or $Q_{ij} = P_{i-j, i-j+1, \ldots, i}$

Example:

$$Q_{3,2}(x) = P_{1,2,3}(x) = \frac{(x-x_1)P_{2,3}(x) - (x-x_3)P_{1,2}(x)}{x_3 - x_1}$$

$$= \frac{(x-x_1)Q_{31} - (x-x_3)Q_{21}}{x_3 - x_1}$$

$Q_{2,1}(x) = ?$

$$Q_{21}(x) = \frac{(x-x_1)P_2(x) - (x-x_2)P_1(x)}{x_2 - x_1}$$

$$= \frac{(x-x_1)Q_{2,0} - (x-x_2)Q_{1,0}}{x_2 - x_1}$$

N_5

$$Q_{21}(x) = \frac{(x-x_1)\,\overset{P_2(x)}{\overset{\uparrow}{Q_{20}(x)}} - (x-x_2)\,\overset{P_1(x)}{\overset{\uparrow}{Q_{10}(x)}}}{x_2 - x_1}$$

$\downarrow$

$P_{1,2}(x)$

In general,

$$Q_{ij}(x) = \frac{(x-x_{i-j})\,\overset{P_{i-(j-1),\ldots,i}}{\overset{\uparrow}{Q_{i,j-1}(x)}} - (x-x_i)\,\overset{P_{i-j,\ldots,i-1}}{\overset{\uparrow}{Q_{i-1,j-1}}}}{x_i - x_{i-j}}$$

$\downarrow$

$$P_{i-j,\, i-(j-1),\ldots,i} = \frac{(x-x_{i-j})\,P_{i-(j-1),\ldots i}(x) - (x-x_i)\,P_{i-j,\ldots,i}(x)}{x_i - x_{i-j}}$$

Implementation: Example 5 (in book)

DATA

	x_0	x_1	x_2	x_3	x_4
x	1	1.3	1.6	1.9	2.2
$f(x)$	0.7651977	0.6200860	0.4554022	0.2818186	0.1103623

Approximate $f(1.5)$ using Lagrange interpolating polynomials.

N_S'

$x - x_i$

$x-x_0 = 1.5 - 1 = 0.5$	$0.76\ldots$				
$x-x_1 = 1.5 - 1.3 = 0.2$	$0.62\ldots$	Q_{11}			
$x-x_2 = 1.5 - 1.6 = -0.1$	$0.45\ldots$	Q_{21}	Q_{22}		
$x-x_3 = 1.5 - 1.9 = -0.4$	$0.28\ldots$	Q_{31}	Q_{32}	Q_{33}	
$x-x_4 = 1.5 - 2.2 = -0.7$	$0.11\ldots$	Q_{41}	Q_{42}	Q_{43}	Q_{44}

$$Q_{11}^{(1.5)} = \frac{(x-x_0)Q_{10} - (x-x_1)Q_{00}}{x_1 - x_0}$$

$$= \frac{(0.5)\,0.62 - (0.2)\,0.76}{0.3} \simeq 0.5233449$$

$$Q_{21}^{(1.5)} = \frac{\overset{(x-x_1)}{(0.2)}\,\overset{Q_{21}}{(0.45\ldots)} - \overset{(x-x_2)}{(-0.1)}\,\overset{Q_{11}}{0.62\ldots}}{+0.3 = (x_2 - x_1)} \simeq 0.5102968$$

$$Q_{22}(1.5) = \frac{\overset{(x-x_0)}{0.5}\,\overset{Q_{21}}{(0.51\ldots)} - \overset{(x-x_2)}{(-0.1)}\,\overset{Q_{22}}{(0.52\ldots)}}{\underset{x_2 - x_0}{0.6}} \simeq 0.5124\ldots$$

Show table 3.4.

$Q_{00} = 0.7651977, \qquad x_0 = 1$

$Q_{10} = 0.6200860, \qquad x_1 = 1.3$

then,

$$Q_{11}(x) = \frac{(x-x_0)\,Q_{1,0}(x) - (x-x_1)\,Q_{0,0}(x)}{x_1 - x_0}$$

$$\Rightarrow \quad Q_{11}(1.5) = \frac{0.5 * 0.6200860 - 0.2 \times 0.7651977}{0.3}$$

$$= 0.5233449.$$

$Q_{21}(1.5)$ uses $Q_{1,0}(1.5)$ and $Q_{2,0}(1.5)$

$\vdots$

$Q_{44}(1.5)$ uses $Q_{33}(1.5)$ and $Q_{4,3}(1.5)$

Show Error interpolation using Lagrange Polynomials.

Application to Example 6.

	x_0	x_1	x_2
$f(x)$	2	2.2	2.3
$f(x)$	0.6931	0.7885	0.8329

$f(x) = \ln(x)$

$f'(x) = 1/x$

$f''(x) = -1/x^2$

$f'''(x) = 2/x^3$

$\Rightarrow \; |f(2.1) - P_2(2.1)| =$

$= \left| \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2) \right|$

$= \left| \frac{1}{3\xi^3}(0.1)(-0.1)(-0.2) \right| \le 8.3 \times 10^{-5}.$

For sintaxis in programming language, we need to change index. For instance,

$$\begin{array}{lll} Q_{00} & & \\ Q_{10} & Q_{11} & \\ Q_{20} & Q_{21} & Q_{22} \end{array} \longrightarrow \begin{array}{lll} Q_{11} & & \\ Q_{21} & Q_{22} & \\ Q_{31} & Q_{32} & Q_{33} \end{array}$$

Algorithm

old

$$\begin{cases} i = 1, 2, .., n \\ \quad j = 1, 2.., i \\ \quad\quad Q_{ij} = \dfrac{(x - x_{i-j})\, Q_{i,j-1} - (x - x_i)\, Q_{i-1,j-1}}{x_i - x_{i-j}} \end{cases} \quad \text{(old)}$$

Ask students to find it

New

$$\begin{cases} i = 2, \dots n \\ \quad j = 2,.., i \\ \quad\quad Q_{ij} = \dfrac{(x - x_{i-(j-1)})\, Q_{i,j-1} - (x - x_i)\, Q_{i-1,j-1}}{x_i - x_{i-(j-1)}} \end{cases}$$

N_6^2

old $$Q_{22} = \frac{(x-x_0)\,Q_{21} - (x-x_2)\,Q_{11}}{x_2 - x_0}$$

New $$Q_{33} = \frac{(x-x_1)\,Q_{32} - (x-x_3)\,Q_{22}}{x_3 - x_1}$$

x_1 ✓	Q_{11} ✓		
x_2	Q_{21}	Q_{22}	
x_3	Q_{31}	Q_{32}	Q_{33}

New $$Q_{22} = \frac{(x-x_1)\,Q_{21} - (x-x_2)\,Q_{11}}{x_2 - x_1}$$

$$Q_{32} = \frac{(x-x_2)\,Q_{31} - (x-x_3)\,Q_{21}}{x_3 - x_2}$$

$i = 2, \ldots, n$
$j = 2, \ldots, i$

$$Q_{ij} = \frac{(x-x_{i-j+1})\,Q_{i,j-1} - (x-x_i)\,Q_{i-1,j-1}}{x_i - x_{i-j+1}}$$